

INEXTENSIBLE DOMAINS AND DOMAINS WITH A CIRCLE OF OUTER BILLIARD TRIANGLES

YOAV KALLUS

ABSTRACT. We develop a theory of planar, origin-symmetric, convex domains that are inextensible with respect to lattice covering, that is, domains such that augmenting them in any way allows fewer domains to cover the same area. We show that origin-symmetric inextensible domains are exactly the origin-symmetric convex domains with a circle of outer billiard triangles. We address a conjecture by Genin and Tabachnikov about convex domains, not necessarily symmetric, with a circle of outer billiard triangles, and show that it follows immediately from a result of L. Fejes Tóth related to the covering problem.

In a series of papers from 1946 to 1947, Kurt Mahler developed a theory of planar, origin-symmetric, star-like domains that are irreducible with respect to lattice packing, that is, domains such that taking away any piece allows more domains to be packed in the same area [5, 6]. In particular, he was interested in the problem of identifying convex domains such that every convex domain with lower area may be packed at a greater number per unit area than the domain in question. This is the subject of Reinhardt's conjecture [7]. In particular, Mahler showed that the disk is not such a domain, even if the domains of lower area to which we compare it are restricted to a small neighborhood of the disk with respect to Hausdorff distance [6].

Inspired by the work of Mahler, we develop a theory of planar, origin-symmetric, convex domains that are inextensible with respect to lattice covering, that is, domains such that adding any piece allows fewer domains to cover the same area. We find that the inextensible domains are simply those with a circle of outer billiard triangles, a family of domains studied previously by Genin and Tabachnikov [3]. The analogue of Reinhardt's conjecture for covering was solved by L. Fejes Tóth who showed that every origin-symmetric convex domain can cover the plane with no larger number per unit area than an ellipse of the same area [8]. In other words, the ellipse covers the plane with the least efficiency.

Genin and Tabachnikov conjecture that out of convex domains, not necessarily symmetric, with a circle of critical triangles of a fixed area, the ellipse provides the upper bound for the area of the domain [3]. A theorem of Fejes Tóth which he uses to prove that the ellipse is the least efficient covering shape yields this conjecture immediately.

We call K a symmetric convex domain (below just “domain”), if K is a convex compact subset of \mathbb{R}^2 such that $K = -K$. A lattice $\Lambda = B\mathbb{Z}^2$ is the image of the integer lattice \mathbb{Z}^2 under a nonsingular linear map B . The determinant $d(\Lambda)$ of a lattice $\Lambda = B\mathbb{Z}^2$ is given by $|\det(B)|$ and is independent of the basis B used. We will use the Hausdorff metric as a distance between domains: $\delta(K, K') = \min\{\varepsilon : K \subseteq K' + \varepsilon D, K' \subseteq K + \varepsilon D\}$, where D is the unit disk. As the distance between lattices we use the distance between the closest two bases: $\delta(\Lambda, \Lambda') = \min\{\|B - B'\| : \Lambda = B\mathbb{Z}^2, \Lambda' = B'\mathbb{Z}^2\}$, where $\|\cdot\|$ is the Hilbert-Schmidt norm.

The lattice Λ is called K -covering if $K + \Lambda = \mathbb{R}^2$. Of all K -covering lattices there is at least one lattice Λ_c such that $d(\Lambda) \leq d(\Lambda_c)$ whenever Λ is K -covering [4]. We call such a lattice a critical lattice for K and its determinant is called the critical determinant of K and denoted $\Delta(K)$. Whenever $K' \supseteq K$, any critical lattice of K is also K' -covering, and therefore $\Delta(K') \geq \Delta(K)$. We say that K is extensible if there is a domain K' containing K but different from it that has the same critical determinant as K . Otherwise, we say K is inextensible.

Let us begin with an example of an inextensible domain: a parallelogram. The critical determinant of a parallelogram is equal to its area. Let K be a parallelogram and let K' be a proper superdomain. By definition there must be a point $\mathbf{x} \in K' \setminus K$. Let $K'' = \text{conv}(K, \pm \mathbf{x})$. Since K'' is either a parallelogram or a hexagon, its critical determinant is equal to its area and therefore $\Delta(K) < \Delta(K'') \leq \Delta(K')$.

Bambah and Rogers have showed that if T is a triangle inscribed in K , maximizing the area among all triangles inscribed in K , then the critical determinant of a domain K is equal to twice the area of T [1]. We call such a triangle a critical triangle of K . Consequently any lattice which is critical for the (possibly degenerate) hexagon $\text{conv}(T, -T)$ is critical for K . Any critical triangle T must include the origin, otherwise a reflection of one of its vertices through the opposite side yields a triangle of equal area but with one vertex in the interior of K . When K is a parallelogram, T may have the origin on its perimeter. When K is not a parallelogram, the origin must be in the interior of T , since otherwise the hexagon $\text{conv}(T, -T)$ degenerates to a parallelogram which is simultaneously a proper subdomain of K and of equal critical determinant, contradicting the inextensibility of parallelograms. Similarly,

the hexagon $\text{conv}(T, -T)$ must also not degenerate to a parallelogram by having one of the vertices of T lie on a side of $-T$ and therefore the relative interiors of all sides of T must be interior to K .

Let $\mathbf{u}_\theta = (\cos \theta, \sin \theta)$ be a point on the unit circle at angle θ from the x -axis. The support height of a domain K in the direction \mathbf{u}_θ is given by $h(\theta) = \max_{\mathbf{x} \in K} \langle \mathbf{x}, \mathbf{u}_\theta \rangle$. The support line $L(\theta)$ is the line $\{\mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{x}, \mathbf{u}_\theta \rangle = h(\theta)\}$, and it intersects the boundary of K but not its interior.

Lemma 1. *Let $L(\theta)$ be a support line of K . Consider all triangles \mathbf{xyz} inscribed in K in such a way that $\mathbf{x} \in L(\theta)$, and the side \mathbf{yz} is parallel to $L(\theta)$ (for definiteness, let \mathbf{xyz} be arranged counter-clockwise). There are points \mathbf{y}_0 and \mathbf{z}_0 such that a triangle of the type described maximizes the area amongst all such triangles if and only if $\mathbf{y} = \mathbf{y}_0$ and $\mathbf{z} = \mathbf{z}_0$.*

Proof. Let \mathbf{xyz} and $\mathbf{x}'\mathbf{y}'\mathbf{z}'$ be two triangles maximizing the area. First note that the area does not depend on the point lying on $L(\theta)$. Therefore, we may take $\mathbf{x} = \mathbf{x}'$ without loss of generality. If the altitudes taken from \mathbf{x} of the two triangles are not equal, then convexity guarantees that any inscribed triangle of the type described whose altitude is between the two altitudes must have strictly greater area. Therefore, the altitudes must be equal, and consequently also the length of the side opposite \mathbf{x} . If the sides opposite \mathbf{x} are not identical, then their convex hull gives the side of an inscribed triangle of the type described with greater area. Therefore, the sides opposite \mathbf{x} must be identical. \square

We denote by $A(\theta)$ the maximal area of a triangle of the type described in Lemma 1, and call a triangle achieving this area a triangle anchored at $L(\theta)$ (or simply at angle θ). Clearly, the set of all triangles anchored at support lines of K contains all the critical triangles of K and $\Delta(K) = 2 \max_{0 \leq \theta < 2\pi} A(\theta) = 2A_{\max}$. Note that any critical triangle occurs as a triangle anchored at three angles (parallel to its three sides) and its symmetric image occurs as a triangle anchored at three more angles. We label these angles θ_i , $i = 1, \dots, 6$, such that $\theta_1 < \theta_2 < \dots < \theta_6 < \theta_1 + 2\pi$ and $\theta_{i+3} = \theta_i + \pi$ (the label i is understood to extend cyclically such that $\theta_{i+6} = \theta_i + 2\pi$). Therefore, we associate each critical triangle paired with its symmetric image (for short, critical triangle pair) with the corresponding angles $\theta_1, \dots, \theta_6$ at which they are anchored. We now show that the angles associated with two critical triangle pairs intersperse.

Lemma 2. *Let θ_1, θ_3 , and θ_5 and θ'_1, θ'_3 , and θ'_5 be the angles at which two distinct critical triangle of K arise, then there is some even integer k such that $\theta_1 \leq \theta'_{k+1} \leq \theta_3 \leq \theta'_{k+3} \leq \theta_5 \leq \theta'_{k+5} \leq \theta_1 + 2\pi$.*

Proof. First let us reduce the problem to the case where all six angles and all six triangle vertices are distinct. If the two triangles are anchored at a common support line, then by Lemma 1 they also have two vertices in common, and the two remaining vertices lie on a line parallel to the line connecting the shared vertices. The lemma in this case then follows easily. Therefore, assume that all six angles are distinct. If the triangles have a vertex in common, then by convexity of K and by the fact that the interior of the triangles must intersect (both must contain the origin) we conclude that the sides opposite the shared vertex must intersect. It follows easily that starting from the shared vertex and traveling counterclockwise along the boundary of K , we meet the vertices of the two triangles alternately and the lemma holds. We assume therefore that the six angles are distinct as are the six vertices. Furthermore, if K is a parallelogram then the lemma follows easily, so we may also assume that K is not a parallelogram and there is no line segment in the boundary of K containing three of the six vertices.

For a proof by contradiction, assume that one of the three intervals (θ_1, θ_3) , (θ_3, θ_5) , and $(\theta_5, \theta_1 + 2\pi)$ includes at least two of the angles θ'_1, θ'_3 , and θ'_5 . Since the two triangles must intersect, it includes exactly two. Without loss of generality then, we may take $\theta_1 \leq \theta'_1 \leq \theta'_3 \leq \theta_3 \leq \theta'_5 \leq \theta_5 \leq \theta_1 + 2\pi$. Denote by \mathbf{x}, \mathbf{y} , and \mathbf{z} the vertices incident on the support lines $L(\theta_1), L(\theta_3)$, and $L(\theta_5)$ respectively, and similarly denote by \mathbf{x}', \mathbf{y}' , and \mathbf{z}' the vertices of the other critical triangle.

We make use of the fact that if $ABCDE$ is a non-degenerate convex pentagon, and $\text{area}(ABD) \leq \text{area}(ABE)$ then $\text{area}(ACD) < \text{area}(ACE)$. This fact is proved, for example, in Ref. [2] under the name of the Pentagon Lemma. Using the Pentagon Lemma and the fact that $\text{area}(\mathbf{x}'\mathbf{y}\mathbf{z}) \leq \text{area}(\mathbf{x}\mathbf{y}\mathbf{z})$ we have that $\text{area}(\mathbf{x}'\mathbf{y}\mathbf{z}') < \text{area}(\mathbf{x}\mathbf{y}\mathbf{z}')$. On the other hand, the fact that $\text{area}(\mathbf{x}\mathbf{y}'\mathbf{z}') \leq \text{area}(\mathbf{x}'\mathbf{y}'\mathbf{z}')$ gives through the Pentagon Lemma that $\text{area}(\mathbf{x}\mathbf{y}\mathbf{z}') < \text{area}(\mathbf{x}'\mathbf{y}\mathbf{z}')$. The two inequalities contradict. \square

Corollary 1. *The angles at which two distinct critical triangle pairs are anchored may be labeled in such a way that $\theta_1 \leq \theta'_1 \leq \theta_2 \leq \theta'_2 \leq \theta_3 \leq \dots \leq \theta_6 \leq \theta'_6 \leq \theta_1 + 2\pi$.*

Intuitively, a domain can only be inextensible if it has a one parameter family of critical lattices so that any augmentation of the domain would allow one of the critical lattices to increase in determinant. We prove this now.

Theorem 1. *K is inextensible if and only if $A(\theta)$ is constant.*

Proof. First assume that K is extensible. Then there exists a proper superdomain $K' \supset K$ such that $\max A'(\theta) = \max A(\theta)$. There exists a support line $L(\theta_0)$ of K which intersects the interior of K' , and consequently $A(\theta_0) < A'(\theta_0) \leq \max A'(\theta) = \max A(\theta)$. Therefore $A(\theta)$ is not constant.

For the converse assume that $A(\theta)$ is not constant. Note that $A(\theta)$ must be continuous in θ , so the set of angles such that $A(\theta) = A_{\max}$ is closed. Consider those support lines $L(\theta)$ of K that are not tangent to K on either side of their intersection with the boundary, that is, support lines such that support lines for sufficiently nearby angles have the same intersection with the boundary. Call support lines of this type non-tangent and call all other support lines tangent. Assume that there is an angle θ such that $L(\theta)$ is tangent and $A(\theta) < A_{\max}$. If such an angle exists, then we may pick it in such a way that all sufficiently nearby angles also have tangent support lines. We can then construct superdomains $K'_\varepsilon \supset K$ with the same support lines as K for angles outside an arbitrarily small neighborhood of θ and such that the support lines for angles inside the neighborhood are moved by arbitrarily small distances. It is clear that $\Delta(K'_\varepsilon) = \Delta(K)$ for small enough ε and K is extensible. Otherwise, there must be for all small enough ε a critical triangle of K'_ε of area greater than A_{\max} and with a vertex on the part of the boundary of K'_ε that is not shared with K . The existence of these triangles would imply the existence of an anchored triangle for K anchored at $L(\theta)$ with area at least A_{\max} , and is therefore impossible.

Suppose on the other hand that all angles for which $A(\theta)$ is submaximal have support lines that are non-tangent. Let us consider an angle θ for which $A(\theta)$ is submaximal and denote as \mathbf{x} the point at which $L(\theta)$ intersects ∂K . By continuity, θ must be part of an open interval (θ_1, θ'_1) such that $A(\theta_1) = A(\theta'_1) = A_{\max}$ and $A(\theta') < A_{\max}$ for all $\theta' \in (\theta_1, \theta'_1)$. Since all angles $\theta' \in (\theta_1, \theta'_1)$ are submaximal, their support lines are non-tangent, and therefore both $L(\theta_1)$ and $L(\theta'_1)$ contain $\mathbf{x} \in \partial K$, and \mathbf{x} is a vertex of two critical triangles anchored at $L(\theta_1)$ and $L(\theta'_1)$. Denote the other two angles anchoring each of these triangles as θ_3 and θ_5 and θ'_3 and θ'_5 . Consider the intervals (θ_3, θ'_3) and (θ_5, θ'_5) . By the interspersing property (Lemma 2), $A(\theta)$ is submaximal at all angles of these intervals, therefore all angles of these intervals must have non-tangent support lines. It follows that all vertices of the two critical triangles coincide. Since this is a contradiction, the case where all submaximal angles have non-tangent support lines is impossible. \square

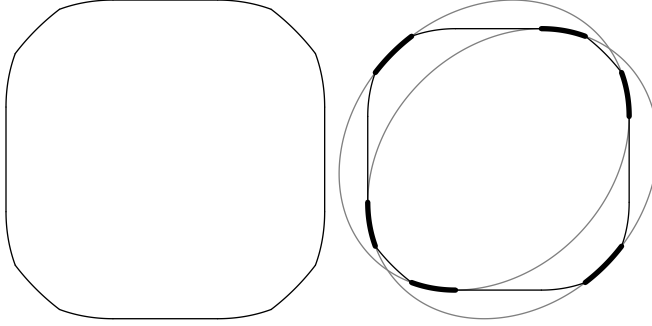


FIGURE 1. An example of an inextensible domain (left). The boundary of the domain is composed of 4 line segments and 12 arcs belonging to 4 ellipses. The three arcs belonging to each ellipse are traced out by the vertices of critical triangles of that ellipse (right). This construction yields a one-parameter family of inextensible domains interpolating between the disk and the square.

Genin and Tabachnikov have observed that convex domains, not necessarily symmetric, such that each support line anchors a critical triangle can be characterized by the equivalent property that each point on its boundary lies on an outer billiard triangle, that is, a circumscribed triangle such that the midpoint of each side is on the boundary of the domain [3]. For terseness, we will say that a domain (again, not necessary symmetric) that possesses these equivalent properties has a circle of critical triangles. The theorem identifies the family of inextensible symmetric convex domains with the family of symmetric convex domains with a circle of critical triangles. Examples of inextensible domains include, in addition to the parallelogram and ellipse, the regular $6n + 4$ -gons. Another example of a family of inextensible domains interpolating between the disk and the square is given in Figure 1.

Genin and Tabachnikov conjecture that, among convex domains with a circle of critical triangles of a given area, the ellipse maximizes the area of the domain. This in fact follows easily from the following theorem of L. Fejes Tóth:

Theorem 2. (*L. Fejes Tóth [8]*) *Let K be a convex domain and T_n the inscribed n -gon of maximal area in K , then*

$$\frac{\text{area } T_n}{\text{area } K} \geq \frac{n}{2\pi} \sin \frac{2\pi}{n}$$

with equality for, and only for, an ellipse.

Corollary 2. *Let K be a convex domain with a circle of critical triangles of area A , then $\text{area } K \leq 4\pi A/\sqrt{27}$, with equality for, and only for, an ellipse.*

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YOAV KALLUS, CENTER FOR THEORETICAL SCIENCES, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544